ICIC Express Letters
Volume **15**, Number **12**, December **2021**

ICIC International ©2021 ISSN 1881-803X
pp. **1291–1298**

ON LEFT AND RIGHT BASES OF AN ORDERED Γ-SEMIHYPERGROUP

Wichayaporn Jantanan[1], Varunee Thongduang[1], Benjamas Saoree[1]
Ronnason Chinram[2] and Pattarawan Petchkaew[3,*]

[1]Department of Mathematics
Faculty of Science
Buriram Rajabhat University
439, Jira Road, Naimuang Sub-district, Muang District, Buriram 31000, Thailand
wichayaporn.jan@bru.ac.th; { vearn23; saoree1233 }@gmail.com

[2]Division of Computational Science
Faculty of Science
Prince of Songkla University
15, Karnjanavanich Road, Hat Yai, Songkhla 90110, Thailand
ronnason.c@psu.ac.th

[3]Mathematics Program
Faculty of Science and Technology
Songkhla Rajabhat University
160, Moo 4, Tambon Khoa-Roob-Chang, Muang District, Songkhla 90000, Thailand
*Cooresponding author: pattarawan.pe@skru.ac.th

Received April 2021; accepted June 2021

Abstract. *The main purpose of this paper is to extend the concepts of left and right bases in* Γ*-semigroups which had been introduced by Changphas and Kummoon in 2018. In this paper, we introduce the concepts of left and right bases of ordered* Γ*-semihypergroups and extend the results in* Γ*-semigroups to ordered* Γ*-semihypergroups. In addition, we investigate the structure of an ordered* Γ*-semihypergroup containing the right bases. We prove that all of the right bases of an ordered* Γ*-semihypergroup have the same cardinality. Furthermore, we also prove that an ordered* Γ*-semihypergroup eliminating the union of all right bases, if it is non-empty, is a left* Γ*-hyperideal of such ordered* Γ*-semihypergroup.*
Keywords: Ordered Γ-semihypergroups, Left (right) Γ-hyperideals, Left (right) bases

1. **Introduction.** Hyperstructure theory was first studied in 1934 by a French mathematician Marty [1]. He introduced the hypergroups, analyzed their properties and applied them to groups and rational algebraic functions. The algebraic hyperstructures are the generalizations of classical algebraic structures. Now, hyperstructure theory is widely studied in many algebraic structures. In this paper, we focus on studying in ordered Γ-semihypergroups which are the generalizations of Γ-semihypergroups, ordered semihypergroups, Γ-semigroups and ordered semigroups. In 2018, Changphas and Kummoon [2] introduced the notion of left and right bases of Γ-semigroups. The structure of a Γ-semigroup containing right bases was studied. They proved that the right bases of a Γ-semigroup have the same cardinality. Moreover, they showed the compliment of the union of all right bases of a Γ-semigroup, if it is non-empty, is a left ideal of the Γ-semigroup. This is a motivation of this paper. The purpose of this paper is to introduce the concept of left and right bases of ordered Γ-semihypergroups and extend the results in Γ-semigroups to ordered Γ-semihypergroups.

DOI: 10.24507/icicel.15.12.1291

2. **Preliminaries.** In this section, we recall the concept of an ordered Γ-semihypergroup and recall some definitions that will be used in this paper.

Definition 2.1. *Let H be a non-empty set and $P^*(H)$ be the family of all non-empty set of H. A mapping $\circ : H \times H \to P^*(H)$ is called a* hyperoperation *on H. If A and B are two non-empty subsets of H and $x \in H$, then we denote*

$$A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Anvariyeh et al. [3] introduced the notion of Γ-semihypergroups as a generalization of semigroups, semihypergroups, and Γ-semigroups.

Definition 2.2. [3] *Let H and Γ be two non-empty sets. H is called a* Γ-semihypergroup *if every $\gamma \in \Gamma$ is a hyperoperation on H, $x\gamma y \subseteq H$ for every $x, y \in H$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in H$ we have $x\alpha(y\beta z) = (x\alpha y)\beta z$.*

Let A and B be two non-empty subsets of H and $\gamma \in \Gamma$. Then define

$$A\gamma B = \bigcup_{a\in A,\, b\in B} a\gamma b \text{ and } A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B.$$

Kondo and Lekkoksung first considered studying ordered Γ-semihypergroups [4]. Later, some remarkable properties of ordered Γ-semihypergroups were studied [5, 6, 7, 8, 9, 10].

Definition 2.3. [4] *$(H, \Gamma, \leq)$ is called an* ordered Γ-semihypergroup *if (H, Γ) is a Γ-semihypergroup and $(H, \leq)$ is a partially ordered set such that the monotone condition holds as follows: $x \leq y \Rightarrow a\gamma x \leq a\gamma y$ and $x\gamma a \leq y\gamma a$, for all $x, y \in H$, $\gamma \in \Gamma$, where, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.*

In the following, we denote an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ by H unless otherwise specified.

Example 2.1. [7] *Let $H = \{x, y, z, r, s, t\}$ and let $\Gamma = \{\gamma, \beta\}$ be the set of binary hyperoperations defined as follows:*

γ	x	y	z	r	s	t
x	$\{x,y\}$	$\{x,y\}$	$\{z\}$	$\{r,s\}$	$\{r,s\}$	$\{t\}$
y	$\{x,y\}$	$\{y\}$	$\{z\}$	$\{r,s\}$	$\{s\}$	$\{t\}$
z	$\{z\}$	$\{z\}$	$\{z\}$	$\{t\}$	$\{t\}$	$\{t\}$
r	$\{r,s\}$	$\{r,s\}$	$\{t\}$	$\{x,y\}$	$\{x,y\}$	$\{z\}$
s	$\{r,s\}$	$\{s\}$	$\{t\}$	$\{x,y\}$	$\{y\}$	$\{z\}$
t	$\{t\}$	$\{t\}$	$\{t\}$	$\{z\}$	$\{z\}$	$\{z\}$

β	x	y	z	r	s	t
x	$\{x,r\}$	$\{x,y,r,s\}$	$\{z,t\}$	$\{x,r\}$	$\{x,y,r,s\}$	$\{z,t\}$
y	$\{x,y,r,s\}$	$\{x,y,r,s\}$	$\{z,t\}$	$\{x,y,r,s\}$	$\{x,y,r,s\}$	$\{z,t\}$
z	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$
r	$\{x,r\}$	$\{x,y,r,s\}$	$\{z,t\}$	$\{x,r\}$	$\{x,y,r,s\}$	$\{z,t\}$
s	$\{x,y,r,s\}$	$\{x,y,r,s\}$	$\{z,t\}$	$\{x,y,r,s\}$	$\{x,y,r,s\}$	$\{z,t\}$
t	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$	$\{z,t\}$

We obtain H is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by

$$\leq := \{(x, x), (x, y), (y, y), (r, r), (r, s), (s, s), (t, t)\}.$$

Definition 2.4. [4] *A nonempty subset A of an ordered Γ-semihypergroup H is called a* sub Γ-semihypergroup *of H if $A\Gamma A \subseteq A$.*

Definition 2.5. [4] *A non-empty subset A of an ordered Γ-semihypergroup H is called a* left (resp. right) Γ-hyperideal *of H if $H\Gamma A \subseteq A$ (resp. $A\Gamma H \subseteq A$) and $a \in A$, $b \leq a$ for $b \in H$ implies $b \in A$. A is called a* two-side Γ-hyperideal *(or simple called a* Γ-hyperideal*) of H if A is both left and right hyperideal of H.*

Proposition 2.1. *Let H be an ordered Γ-semihypergroup and B_i be a left Γ-hyperideal of H for each $i \in I$. Then, the following assertions hold:*

1) if $\bigcap_{i\in I} B_i \neq \emptyset$ then $\bigcap_{i\in I} B_i$ is a left Γ-hyperideal of H.

2) $\bigcup_{i\in I} B_i$ is a left Γ-hyperideal of H.

Let K be a non-empty subset of an ordered Γ-semihypergroup H. We define $(K] := \{x \in H \mid x \leq k \text{ for some } k \in K\}$, for $K = \{k\}$, we write $(k]$ instead of $(\{k\}]$. If A and B are non-empty subsets of H, then we have

1) $A \subseteq (A]$;
2) $((A]] = (A]$;
3) If $A \subseteq B$, then $(A] \subseteq (B]$;
4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;
5) $((A]\Gamma(B]] = (A\Gamma B]$;
6) $(A \cup B] = (A] \cup (B]$.

Definition 2.6. [7] *A proper left Γ-hyperideal M of an ordered Γ-semihypergroup H, $(M \neq H)$ is said to be* maximal *if for any left Γ-hyperideal A of $H, M \subseteq A \subseteq H$ implies $M = A$ or $A = H$.*

Definition 2.7. [8] *Let A be a non-empty subset of an ordered Γ-semihypergroup H. Then intersection of all left Γ-hyperideals of H containing A is the* smallest left Γ-hyperideal of H generated by A *and denoted by $L(A)$.*

Lemma 2.1. [8] *Let A be a non-empty subset of an ordered Γ-semihypergroup H. Then $L(A) = (A \cup H\Gamma A]$.*

Corollary 2.1. [8] *Let a be an element of an ordered Γ-semihypergroup H. Then $L(a) = (a \cup H\Gamma a]$.*

3. **Definitions and Lemmas.**

In this section, we give some definitions and lemmas that will be used in this paper.

Definition 3.1. *Let H be an ordered Γ-semihypergroup. A non-empty subset A of H is called a* right base *of H if it satisfies the following two conditions:*

1) $H = (A \cup H\Gamma A]$, i.e., $H = L(A)$;

2) if B is a subset of A such that $H = L(B)$, then $B = A$.

The definition of a *left base* of H is dually defined.

Example 3.1. *Let $H = \{a, b, c, d\}$ and let $\Gamma = \{\gamma, \beta\}$ be the set of binary hyperoperations defined as follows:*

γ	a	b	c	d
a	$\{a\}$	$\{b,d\}$	$\{c\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$
c	$\{c\}$	$\{b,d\}$	$\{a\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

β	a	b	c	d
a	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$
c	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

and $\leq := \{(a,a),(a,c),(b,b),(b,d),(c,c),(d,d)\}$.

Then H is an ordered Γ-semihypergroup. The right bases of H are $A_1 = \{a\}$ and $A_2 = \{c\}$ but $A_3 = \{a, c\}$ and $A_4 = \{b, d\}$ are not the right bases of H. The left bases of H are the same as the right bases of H.

Example 3.2. *Let $H = \{a, b, c, d, e, f\}$ and let $\Gamma = \{\gamma, \beta\}$ be the set of binary hyperoperations defined as follows:*

γ	a	b	c	d	e	f
a	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{a\}$	$\{b\}$	$\{a, c\}$	$\{a\}$	$\{a\}$	$\{a, f\}$
d	$\{a\}$	$\{b\}$	$\{a, e\}$	$\{a\}$	$\{a\}$	$\{a, d\}$
e	$\{a\}$	$\{b\}$	$\{a, e\}$	$\{a\}$	$\{a\}$	$\{a, d\}$
f	$\{a\}$	$\{b\}$	$\{a, c\}$	$\{a\}$	$\{a\}$	$\{a, f\}$

β	a	b	c	d	e	f
a	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{b\}$	$\{a\}$	$\{a, d\}$	$\{a, e\}$	$\{a\}$
e	$\{a\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
f	$\{a\}$	$\{b\}$	$\{a\}$	$\{a, f\}$	$\{a, c\}$	$\{a\}$

$$\text{and } \leq := \{(a, a), (a, b), (a, c), (a, d), (a, e), (a, f), (b, b), (c, c), (d, d), (e, e), (f, f)\}.$$

Then H is an ordered Γ-semihypergroup. The right bases of H are $A_1 = \{c, d\}$, $A_2 = \{c, f\}$, $A_3 = \{d, e\}$ and $A_4 = \{e, f\}$ but $A_5 = \{a\}$ and $A_6 = \{b\}$ are not the right bases of H.

Lemma 3.1. *Let A be a right base of an ordered Γ-semihypergroup H and let $a, b \in A$. If $a \in (H\Gamma b]$, then $a = b$.*

Proof: Assume $a, b \in A$ such that $a \in (H\Gamma b]$ and $a \neq b$. Let $B = A \setminus \{a\}$, then $B \subset A$. Since $a \neq b$, we have $b \in B$. We will show that $L(A) \subseteq L(B)$. Let $x \in L(A) = (A \cup H\Gamma A]$. It follows that $x \leq z$ for some $z \in A \cup H\Gamma A$. There are two cases to consider.

Case 1: $z \in A$. If $z \neq a$, then $z \in B \subseteq (B \cup H\Gamma A]$. Since $x \leq z$ and $z \in B \subseteq (B \cup H\Gamma B]$, we obtain $x \in ((B \cup H\Gamma B]] = (B \cup H\Gamma B]$. Hence $x \in L(B)$. If $z = a$, then $z = a \in (H\Gamma B] \subseteq (B \cup H\Gamma B]$. Since $x \leq z$ and $z \in (B \cup H\Gamma B]$, we get $x \in ((B \cup H\Gamma B]] = (B \cup H\Gamma B)$. Therefore, $x \in L(B)$.

Case 2: $z \in H\Gamma A$. Then $z \in s\gamma c$ for some $s \in H$, $\gamma \in \Gamma$ and $c \in A$. If $c = a$, then $z \in s\gamma a \subseteq H\Gamma(H\Gamma b] \subseteq (H]\Gamma(H\Gamma b] \subseteq (H\Gamma(H\Gamma b] \subseteq (H\Gamma b)] \subseteq (B \cup H\Gamma B]$. Since $x \leq z$ and $z \in (B \cup H\Gamma B]$, we have $x \in ((B \cup H\Gamma B]] = (B \cup H\Gamma B)$. So $x \in L(B)$. If $c \neq a$, then $c \in B$. Hence $z \in s\gamma c \subseteq H\Gamma B \subseteq (B \cup H\Gamma B]$. Since $x \leq z$ and $z \in (B \cup H\Gamma B]$, we obtain $x \in ((B \cup H\Gamma B]] = (B \cup H\Gamma B]$. So $x \in L(B)$. Therefore, $L(A) \subseteq L(B)$.

By $H = L(A) \subseteq L(B) \subseteq H$, it follows that $L(B) = H$. This is a contradiction. Hence $a = b$. Therefore, the proof is completed. $\square$

Let H be an ordered Γ-semihypergroup. Define a *quasi-order* $\leq_L$ on H by, for any $a, b \in H$,

$$a \leq_L b \Leftrightarrow L(a) \subseteq L(b).$$

The symbol $a <_L b$ stands for $a \leq_L b$ but $a \neq b$.

Lemma 3.2. *Let H be an ordered Γ-semihypergroup. For any $a, b \in H$, if $a \leq b$, then $a \leq_L b$.*

Proof: Let $a, b \in H$ such that $a \leq b$. We will show that $a \leq_L b$, i.e., $L(a) \subseteq L(b)$. Suppose that $x \in L(a)$. Since $x \in (a \cup H\Gamma a]$, we have $x \leq y$ for some $y \in a \cup H\Gamma a$. Since $y \in a \cup H\Gamma a$, we obtain $y = a$ or $y \in H\Gamma a$. If $y = a$, then $x \leq a \leq b$. Hence $x \leq b$ for some $b \in (b \cup H\Gamma b]$. So $x \in ((b \cup H\Gamma b]] = (b \cup H\Gamma b]$. Thus $x \in L(b)$. If $y \in H\Gamma a$, then $y \in s\gamma a$ for some $s \in H$, $\gamma \in \Gamma$. Then $s\gamma a \leq s\gamma b$ and $s\gamma b \subseteq H\Gamma b \subseteq (b \cup H\Gamma b]$ because $a \leq b$. It implies that $y \in s\gamma a \subseteq ((b \cup H\Gamma b]] = (b \cup H\Gamma b]$. Since $x \leq y$ and $y \in (b \cup H\Gamma b]$, we get $x \in ((b \cup H\Gamma b]] = (b \cup H\Gamma b]$. Then $x \in L(b)$. Hence $L(a) \subseteq L(b)$, i.e., $a \leq_L b$. $\square$

Nevertheless, the converse of Lemma 3.2 is not true in general. By Example 3.1, we have $L(c) \subseteq L(a)$, i.e., $c \leq_L a$ but $c \leq a$ is false.

4. **Results.** In this part, the algebraic structure of an ordered Γ-semihypergroup containing right bases will be presented.

Theorem 4.1. *A non-empty subset A of an ordered Γ-semihypergroup H is right base of H if and only if A satisfies the following two conditions:*
1) for any $x \in H$ there exists $a \in A$ such that $x \leq_L a$;
2) for any two distinct elements $a, b \in A$ neither $a \leq_L b$ nor $b \leq_L a$.

Proof: Assume that A is a right base of H. Clearly, $H = L(A)$. Let $x \in H$. Then $x \in (A \cup H\Gamma A]$. Since $x \in (A \cup H\Gamma A]$, we have $x \leq y$ for some $y \in A \cup H\Gamma A$. Then $y \in A$ or $y \in H\Gamma A$. If $y \in A$, then $x \leq_L y$ by Lemma 3.2 and $x \leq y$. If $y \in H\Gamma A$, then $y \in s\gamma a$ for some $s \in H$, $\gamma \in \Gamma$ and $a \in A$. Hence $y \in H\Gamma a \subseteq (a \cup H\Gamma a]$ and $H\Gamma y \subseteq H\Gamma(H\Gamma a) = (H\Gamma H)\Gamma a \subseteq H\Gamma a \subseteq (a \cup H\Gamma a]$, and so $y \cup H\Gamma y \subseteq (a \cup H\Gamma a]$ and $(y \cup H\Gamma y] \subseteq ((a \cup H\Gamma a]] = (a \cup H\Gamma a]$. This implies that $L(y) \subseteq L(a)$, i.e., $y \leq_L a$. Since $x \leq y$, by Lemma 3.2, we have $x \leq_L y$. So $x \leq_L a$. Hence the condition 1) holds. Let $a, b \in A$ such that $a \neq b$. Suppose $a \leq_L b$. We set $B = A \setminus \{a\}$. Then $b \in B$. Let $x \in H$, by 1), there exists $c \in A$ such that $x \leq_L c$. Since $c \in A$, there are two cases to consider. If $c = a$, then $x \leq_L b$. Hence $x \in L(B)$ since $b \in B$. We have $H = I(B)$. This is a contradiction. If $c \neq a$, then $c \in B$. So we obtain $x \in L(x) \subseteq L(c) \subseteq L(B)$. This means that $H = L(B)$. This is a contradiction. The case $b \leq_L a$ can be proved similarly. Hence the condition 2) holds.

Conversely, assume that the conditions 1) and 2) hold. We will show that $H = L(A)$. Let $x \in H$, by 1), there exists $a \in A$ such that $x \leq_L a$, i.e., $L(x) \subseteq L(a)$. Then $x \in L(x) \subseteq L(a) \subseteq L(A)$. Thus $H \subseteq L(A)$, so we have $H = L(A)$. Next, we will show that A is a minimal subset of H with the property $H = L(A)$. Let $B \subset A$ such that $H = L(B)$. Since $B \subset A$, there exists $a \in A \setminus B$. Then $a \in A \subseteq H = (B \cup H\Gamma B] = (B] \cup (H\Gamma B]$. If $a \in (B]$, then $a \leq y$ for some $y \in B$, and by Lemma 3.2, we have $a \leq_L y$. This is a contradiction. Thus $a \notin (B]$, and so we have $a \in (H\Gamma B]$. Since $a \in (H\Gamma B]$, we obtain $a \leq c$ for some $c \in H\Gamma B$. Let $c \in s\gamma b$ for some $s \in H$, $\gamma \in \Gamma$ and $b \in B$. Since $a \leq c$ and $c \in s\gamma b \subseteq H\Gamma b \subseteq b \cup H\Gamma b$. This implies that $a \in (b \cup H\Gamma b]$, and so $H\Gamma a \subseteq H\Gamma(b \cup H\Gamma b] = (H]\Gamma(b \cup H\Gamma b] \subseteq (H\Gamma(b \cup H\Gamma b)] = (H\Gamma b \cup H\Gamma(H\Gamma b)] = (H\Gamma b \cup (H\Gamma H)\Gamma b] \subseteq (H\Gamma b \cup H\Gamma b] \subseteq (b \cup H\Gamma b]$. Then $a \cup H\Gamma a \subseteq (b \cup H\Gamma b]$. Thus $(a \cup H\Gamma a] \subseteq ((b \cup H\Gamma b]] = (b \cup H\Gamma b]$. Hence $L(a) \subseteq L(b)$, i.e., $a \leq_L b$. This is a contradiction. Therefore, A is right base of H. $\square$

If a right base A of an ordered Γ-semihypergroup H is a left Γ-hyperideal of H, then

$$H = (A \cup H\Gamma A] = (A \cup A] = (A] = A.$$

Hence $H = A$. The converse statement is obvious. Then we conclude that

Theorem 4.2. *A right base A of an ordered Γ-semihypergroup H is a left Γ-hyperideal of H if and only if $A = H$.*

In Example 3.1 and Example 3.2, it is observed that the cardinalities of right bases are the same. However, it turns out that this statment is true in general, and we will prove in the following theorem.

Theorem 4.3. *A right base A of an ordered Γ-semihypergroup H has the same cardinality.*

Proof: Let A and B be right bases of an ordered Γ-semihypergroup H. Let $a \in A$. Since B is a right base of H, by Theorem 4.1 1), there exists $b \in B$ such that $a \leq_L b$. Similarly, since A is a right base of H, there exists $a' \in A$ such that $b \leq_L a'$. This means that $a \leq_L b \leq_L a'$, so $a \leq_L a'$. By Theorem 4.1 2), $a = a'$. Hence $L(a) = L(b)$. Now, define a mapping

$$\varphi : A \to B;\ \varphi(a) = b$$

for all $a \in A$. First, we will show that φ is well-defined, and let $a_1, a_2 \in A$ such that $a_1 = a_2$, $\varphi(a_1) = b_1$, and $\varphi(a_2) = b_2$, for some $b_1, b_2 \in B$. So we have $L(a_1) = L(b_1)$ and $L(a_2) = L(b_2)$. Since $a_1 = a_2$, we obtain $L(a_1) = L(a_2)$. Hence $L(a_1) = L(a_2) = L(b_1) = L(b_2)$. We have $b_1 \leq_L b_2$ and $b_2 \leq_L b_1$, by Theorem 4.1 2), $b_1 = b_2$. Thus $\varphi(a_1) = \varphi(a_2)$. Therefore, φ is well-defined. Next, we will show that φ is one-to-one, and let $a_1, a_2 \in A$ such that $\varphi(a_1) = \varphi(a_2)$. Suppose that $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. Hence $L(a_1) = L(a_2) = L(b)$. Since $L(a_1) = L(a_2)$, we get $a_1 \leq_L a_2$ and $a_2 \leq_L a_1$. Hence $a_1 = a_2$. Therefore, φ is one-to-one. Finally, we will show that φ is onto. Let $b \in B$, and then there exists $a \in A$ such that $b \leq_L a$. Similarly, there exists $b' \in B$ such that $a \leq_L b'$. Then $b \leq_L a \leq_L b'$, i.e., $b \leq_L b'$. By Theorem 4.1 2), $b = b'$. Thus $L(a) = L(b)$. So we have $\varphi(a) = b$. Therefore, φ is onto. This completes the proof. □

Theorem 4.4. *Let A be a right base of an ordered Γ-semihypergroup of H, and let $a \in A$. If $L(a) = L(b)$ for some $b \in H$ such that $a \neq b$, then b is an element of a right base of H which is distinct from A.*

Proof: Assume that $L(a) = L(b)$ for some $b \in H$ such that $a \neq b$. Setting $B = (A \setminus \{a\}) \cup \{b\}$, then $B \neq A$. We will show that B is a right base of H. Now, let $x \in H$. Since A is a right base of H, by Theorem 4.1 1), $x \leq_L c$ for some $c \in A$. If $c \neq a$, then $c \in B$. If $c = a$, then $L(c) = L(a)$. Since $L(a) = L(b)$ and $L(c) = L(a)$, we have $L(c) = L(b)$, i.e., $c \leq_L b$. Hence $x \leq_L c \leq_L b$. Thus $x \leq_L b$ where $b \in B$. Next, let $b_1, b_2 \in B$ such that $b_1 \neq b_2$. There are four cases to consider.

Case 1: $b_1 \neq b$ and $b_2 \neq b$. Then $b_1, b_2 \in A$. Since A is a right base of H, neither $b_1 \leq_L b_2$ nor $b_2 \leq_L b_1$.

Case 2: $b_1 \neq b$ and $b_2 = b$. Then $L(b_2) = L(b)$. If $b_1 \leq_L b_2$, then $L(b_1) \subseteq L(b_2) = L(b) = L(a)$. Thus $b_1 \leq_L a$ because $b_1, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $L(a) = L(b) = L(b_2) \subseteq L(b_1)$. Hence $a \leq_L b_1$ since $b_1, a \in A$. This is a contradiction.

Case 3: $b_1 = b$ and $b_2 \neq b$. Then $L(b_1) = L(b)$. If $b_1 \leq_L b_2$, then $L(a) = L(b) = L(b_1) \subseteq L(b_2)$. Thus $a \leq_L b_2$ where $b_2, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $L(b_2) \subseteq L(b_1) = L(b) = L(a)$. Thus $b_2 \leq_L a$ where $b_2, a \in A$. This is a contradiction.

Case 4: $b_1 = b$ and $b_2 = b$. This is impossible.

Therefore, B is a right base of H which is distinct from A. □

Theorem 4.5. *Let C be the union of all right bases of an ordered Γ-semihypergroup H. If $H \setminus C$ is non-empty, then $H \setminus C$ is a left Γ-hyperideal of H.*

Proof: Assume that $H \setminus C \neq \emptyset$. We will show that $H \setminus C$ is a left Γ-hyperideal of H. First, let $x \in H$, $\gamma \in \Gamma$ and $a \in H \setminus C$. To show that $x\gamma a \subseteq H \setminus C$. Suppose that $x\gamma a \nsubseteq H \setminus C$. So, there exists $y \in x\gamma a$ such that $y \in C$. Thus $y \in A$ for some a right base A of H. We have $y \in x\gamma a \subseteq H\Gamma a \subseteq (a \cup H\Gamma a]$ and $H\Gamma y \subseteq H\Gamma(H\Gamma a) = (H\Gamma H)\Gamma a \subseteq H\Gamma a \subseteq (a \cup H\Gamma a]$. This means that $y \cup H\Gamma y \subseteq (a \cup H\Gamma a]$ and $(y \cup H\Gamma y] \subseteq (a \cup H\Gamma a]$. We obtain $L(y) \subseteq L(a)$. If $L(y) = L(a)$, then $a \in C$ by Theorem 4.1. This is a contradiction. Hence $L(y) \subset L(a)$, i.e., $y <_L a$. Since A is a right base of H, there exists $b' \in A$ such that $a \leq b'$. So we have $y <_L a \leq_L b'$ and $y \leq_L b'$ because $y, b' \in A$. This is a contradiction. Thus $x\gamma a \subseteq H \setminus C$. Next, let $y \in H \setminus C$ and $z \in H$ such that $z \leq y$. To prove that $z \in H \setminus C$. If $z \in C$, then $z \in B$ for a right base B of H. Since B is a right base of H and $y \in H$, there exists $c \in B$ such that $y \leq_L c$. Since $z \leq y$, by Lemma 3.1, we have $z \leq_L y$. So $z \leq_L c$ because $z, c \in B$. This is a contradiction. Thus $z \notin C$, i.e., $z \in H \setminus C$. Therefore, $H \setminus C$ is a left Γ-hyperideal of H. □

In Example 3.2, the right bases of H are $A_1 = \{c, f\}$, $A_2 = \{c, d\}$, $A_3 = \{d, e\}$ and $A_4 = \{e, f\}$. Let C be the union of all right bases of H. We obtain $H \setminus C = \{a, b\}$ is a left Γ-hyperideal of H, but it is not a maximal proper left Γ-hyperideal of H. In the following theorem, we provide conditions for $H \setminus C$ to be a maximal proper left Γ-hyperideal of H.

Theorem 4.6. *Let C be the union of all right bases of an ordered Γ-semihypergroup H such that $C \neq \emptyset$. Then $H \setminus C$ is a maximal proper left Γ-hyperideal of H if and only if $C \neq H$ and $C \subseteq L(a)$ for all $a \in C$.*

Proof: Let $H \setminus C$ be a maximal proper left Γ-hyperideal of H. It is clear that $C \neq H$. Let $a \in C$. Suppose $C \nsubseteq L(a)$. Then there exists $b \in C$ such that $b \notin L(a)$. Since $b \notin H \setminus C$, and $b \notin L(a)$, we obtain $(H \setminus C) \cup L(a) \subset H$. Thus $(H \setminus C) \cup L(a)$ is a proper left Γ-hyperideal of H. Hence $H \setminus C \subsetneq (H \setminus C) \cup L(a)$. This contradicts to the maximality of $H \setminus C$. Therefore, $C \subseteq L(a)$ for all $a \in C$.

Conversely, let $C \neq H$ and $C \subseteq L(a)$ for all $a \in C$. Then $C \subset H$, $H \setminus C \subset H$. Since $H \setminus C \neq \emptyset$, by Theorem 4.3, $H \setminus C$ is a proper left Γ-hyperideal of H. Let A be a left Γ-hyperideal of H such that $H \setminus C \subseteq A \subseteq H$. Suppose that $H \setminus C \neq A$. Since $H \setminus C \subset A$, there exists $x \in A$ such that $x \notin H \setminus C$, i.e., $x \in C$. This means that $A \cap C \neq \emptyset$. Let $a \in A \cap C$. Then $a \in A$ and $H\Gamma a \subseteq H\Gamma A \subseteq A$. So $a \cup H\Gamma a \subseteq A$ and $(a \cup H\Gamma a] \subseteq (A] = A$. We obtain $H = (H \setminus C) \cup C \subseteq A \cup L(a) \subseteq A \subseteq H$ because $L(a) \subseteq A$, $C \subseteq L(a)$ and $H \setminus C \subset A$. So $H = A$. Therefore, $H \setminus C$ is a maximal proper left Γ-hyperideal of H. □

In Example 3.1, the right bases of H are $\{a\}$ and $\{c\}$. Let C be the union of all right bases of H. We obtain $H \setminus C = \{b, d\}$. It follows that $C \neq H$ and $C \subseteq L(a)$ for all $a \in C$. Therefore, $H \setminus C = \{b, d\}$ is a maximal proper left Γ-hyperideal of H.

Theorem 4.7. *Let C be the union of all right bases of an ordered Γ-semihypergroup H such that $\emptyset \neq C \subset H$. If H contains a maximal proper left Γ-hyperideal of H, denoted by M^*, then $H \setminus C = M^*$ if and only if $|A| = 1$ for every right base A of H.*

Proof: Assume that $H \setminus C = M^*$. Then $H \setminus C$ is a maximal proper left Γ-hyperideal of H. By Theorem 4.4, $C \subseteq L(a)$ for all $a \in C$. We will show that $H \setminus C \subseteq L(a)$ for all $a \in C$. Suppose that $H \setminus C \nsubseteq L(a')$ for some $a' \in C$. Since $L(a') \subset H$ and $L(a')$ is a proper left Γ-hyperideal of H, it follows that $L(a') \subseteq M^* = H \setminus C$, so we have $a' \in H \setminus C$. This is a contradiction. Hence $H \setminus C \subseteq L(a)$ for all $a \in C$. By $H = (H \setminus C) \cup C \subseteq L(a) \subseteq H$ for all $a \in C$, it implies that $L(a) = H$ for all $a \in C$. Therefore, $\{a\}$ is a right base of H for all $a \in C$. Let A be a right base of H, and let $a, b \in A$. Suppose that $a \neq b$. Since $A \subseteq C$, $a \in C$, we get $H = L(a)$. Since $a \neq b$ and $b \in H = (a \cup H\Gamma a]$, we get $b \in (H\Gamma a]$. By Lemma 3.1, $a = b$. This is a contradiction. Hence $a = b$. Therefore, $|A| = 1$.

Conversely, assume that every right base of H has only one element. Then $H = L(a)$ for all $a \in C$. We will show that $H \setminus C = M^*$. Since $\emptyset \neq H \setminus C \subset H$, by Theorem 4.3, $H \setminus C$ is a proper left Γ-hyperideal of H. Next, let M be a left Γ-hyperideal of H such that $M \nsubseteq H \setminus C$. Hence, there exist $a \in M$ and $a \notin H \setminus C$, i.e., $a \in C$. We have $H\Gamma a \subseteq H\Gamma M \subseteq M$ since $a \in M$. This means that $a \cup H\Gamma a \subseteq M$ and $(a \cup H\Gamma a] \subseteq (M] = M$. Hence $H = (a \cup H\Gamma a] \subseteq M \subseteq H$. This implies that $M = H$. Therefore, $H \setminus C = M^*$. The proof is complete. □

5. **Conclusion and Discussion.** In this paper, we prove that a non-empty subset A of an ordered Γ-semihypergroup H is right base of H if and only if A satisfies the following two conditions: for any $x \in H$ there exists $a \in A$ such that $x \leq_L a$, and for any two distinct elements $a, b \in A$ neither $a \leq_L b$ nor $b \leq_L a$. In addition, we prove that all of the right bases of an ordered Γ-semihypergroup have the same cardinality. Moreover, we also prove that an ordered Γ-semihypergroup eliminating the union of all right bases, if it is non-empty, is a left Γ-hyperideal of such ordered Γ-semihypergroup. Let C be the union of all right bases of an ordered Γ-semihypergroup H such that $\emptyset \neq C \subset H$. We give the conditions for $H \setminus C$ to be a maximal proper left Γ-hyperideal of H. We prove that $H \setminus C$ is a maximal proper left Γ-hyperideal of H if and only if $C \neq H$ and $C \subseteq L(a)$ for all $a \in C$. In the final, we obtained that if an ordered Γ-semihypergroup H contains a maximal proper left Γ-hyperideal of H, denoted by M^*, then $H \setminus C = M^*$ if and only

if $|A| = 1$ for every right base A of H. Moreover, The results in [2] are a special case of this paper.

An ordered Γ-semihypergroup is a generalization of many algebraic structures, for example, Γ-semihypergroups, ordered semihypergroups, Γ-semigroups and ordered semigroups. In the potential future research, the researchers can study directly this structure and apply this results in many algebraic structures or extend the results in some algebraic structures to the results in ordered Γ-semihypergroups. For example in [11], the author studied the picture fuzzy sets in semigroups, we can extend this result to the picture fuzzy sets in Γ-semihypergroups.

REFERENCES

[1] F. Marty, On a generalization of the notion of group, *Proc. of the 8th Congress of Scandinavian Mathematicians*, Stockholm, Sweden, pp.45-49, 1934.

[2] T. Changphas amd P. Kummoon, On left and right bases of a Γ-semigroup, *International Journal of Pure and Applied Mathematics*, vol.118, no.1, pp.125-135, 2018.

[3] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, *Carpathian Journal of Mathematics*, vol.26, no.1, pp.11-23, 2010.

[4] M. Kondo and N. Lekkoksung, On intra-regular ordered Γ-semihypergroups, *International Journal of Mathematical Analysis*, vol.7, no.25, pp.1379-1386, 2013.

[5] M. Y. Abbasi, K. Hila, S. A. Khan and A. F. Talee, Characterizations of soft Γ-hyperideals in ordered Γ-semihypergroups, *Afrika Matematika*, vol.31, nos.5-6, pp.847-867, 2020.

[6] B. Davvaz and S. Omidi, Bi-Γ-hyperideals and Greens relation in ordered Γ-semihypergroup, *Eurasian Mathematical Journal*, vol.8, no.4, pp.63-73, 2017.

[7] B. Davvaz and S. Omidi, Some properties of quasi-Γ-hyperideal and hyperfilters ordered Γ-semihypergroup, *Southeast Asian Bulletin of Mathematics*, vol.42, no.2, pp.223-242, 2018.

[8] B. Davvaz, K. Hila and S. Omidi, Characterizations of regular and intra-regular ordered Γ-semihypergroup in terms of bi-Γ-hyperiddeals, *Carpathian Mathematical Publications*, vol.11, no.1, pp.136-151, 2019.

[9] J. Tang and R. Wang, An investigation on fuzzy Γ-hyperfilters in ordered Γ-semihypergroups, *ICIC Express Letters, Part B: Applications*, vol.7, no.12, pp.2611-2617, 2016.

[10] J. Tang, B. Davvaz, X.-Y. Xie and N. Yaqoob, On fuzzy interior Γ-hyperideals in ordered Γ-semihypergroupsm, *Journal of Intelligent and Fuzzy Systems*, vol.32, no.3, pp.2447-2460, 2017.

[11] P. Yiarayong, Semigroup characterized by picture fuzzy sets, *International Journal of Innovative Computing, Information and Control*, vol.16, no.6, pp.2121-2130, 2020.